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Uninfected random walkers in one dimension

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We consider a system of unbiased diffusing walkers ($A\emptyset \leftrightarrow \emptyset A$) in one dimension with random initial conditions. We investigate numerically the relation between the fraction of walkers $U(t)$ which have never encountered another walker up to time *t*, calling such walkers "uninfected" and the fraction of sites $P(t)$ which have never been visited by a diffusing particle. We extend our study to include the $A+B\rightarrow\emptyset$ diffusion-limited reaction in one dimension, with equal initial densities of *A* and *B* particles distributed homogeneously at $t=0$. We find $U(t)\simeq[P(t)]^{\gamma}$, with $\gamma\simeq1.39$, in both models, though there is evidence that a smaller value of γ is required for $t\rightarrow\infty$.

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I. INTRODUCTION

Random walks model a host of phenomena and find application in virtually all areas of physics [1–5]. In this paper, we consider a one-dimensional (1D) system of noninteracting, unbiased diffusing walkers $A\emptyset \leftrightarrow \emptyset A$, with random initial conditions. In this model each walker undergoes an independent random walk, with multiple occupancy of sites allowed [6]. Recently, we have studied the *persistence* properties of this model [7]. The persistence probability $P(t)$ has been widely studied in recent years [7–20]. In the present context, $P(t)$ is defined as the fraction of sites which have never been visited by a diffusing particle up to time *t*. Such sites are termed "persistent sites." We showed [7] that, in the case of the $A\emptyset \leftrightarrow \emptyset A$ processes

$$P(t)=\exp[-(16\rho_0^2 Dt/\pi)^{1/2}], \tag{1}$$

where ρ_0 is the particle density and D is the diffusion constant. The fraction of unvisited sites in this model can equally be thought of as the surviving fraction of immobile particles in the presence of randomly diffusing traps. At $t=0$, consider each site to contain a static particle *A* so that $\rho_A(0)=1$. Then for $t>0$, let particles of another type *B* randomly distributed on the lattice with density ρ_0, diffuse though the system with the *B* particles acting as perfect traps for the *A* particles so that $A+B\rightarrow B$. This model is just the well-studied "scavenger" model of Redner and Kang, Blumen *et al.*, and Bénichou *et al.* [21–23]. The fraction of unvisited sites is then just $\rho_A(t)$.

The focus of this paper is to explore the relation between $P(t)$ and the fraction of walkers $U(t)$ that have never encountered another walker up to time *t*. To facilitate our discussion, we introduce the following definition. We consider all walkers to be *uninfected* at time $t=0$. The walkers are, however, considered to be mutually infectious, so that for $t>0$ any contact between walkers leads to mutual contamination. We therefore define an uninfected walker to be one which has never encountered (occupied the same site as) another walker. We can then refer to $U(t)$ as the fraction of uninfected walkers. The fraction of uninfected walkers is, in fact, isomorphic to the survival probability of *A* particles in the reaction $A+B\rightarrow B$, with equal mobility for both species [21,24,25,27]. At late times, one expects the asymptotics of this model to be identical to those of the $A+B\rightarrow\emptyset$ diffusion-limited process with $\rho_A(0)\leqslant\rho_B(0)$. For this process it was proved rigorously by Bramson and Lebowitz [25] that

$$\rho_A(t)\sim\exp(-\lambda t^{1/2}) \tag{2}$$

in one dimension, with unknown constant λ. Therefore, we expect identical asymptotic behavior for $U(t)$. The kinetics of the two-species annihilation model has received considerable attention over the past two decades and is largely understood [5,24,25,28–32]. It is well appreciated that in this model, there exists an upper-critical dimension d_c, below which spatial fluctuations in the initial distribution of the reactants play a significant role in the evolution of the density of the particles. This dependence on the microscopic fluctuations invalidates traditional approaches such as the mean-field approximation which yields $\rho(t)\sim t^{-1}$. It was first shown by Toussaint and Wilczek [28] that, for equal initial densities $[\rho_A(0)=\rho_B(0)]$ of randomly diffusing *A* and *B* particles, the walker density decays anomalously, according to $\rho_A(t)\sim\sqrt{\rho_A(0)}t^{-d/4}$ for $d\leqslant d_c=4$, assuming that the particles are distributed homogeneously at $t=0$. In the case of this model, we argued heuristically, on the basis of a toy model [7] that

$$P(t)\sim\exp[-C(\rho_0^2 Dt)^{1/4}], \tag{3}$$

where *C* is a constant and $\rho_0=\rho_A(0)+\rho_B(0)$. We extend our study to include an investigation of the fraction of uninfected walkers in this model in one dimension.

The fraction of uninfected walkers $U(t)$ is related to the fraction of persistent sites $P(t)$ in the following manner. In evaluating $P(t)$, one considers particles with diffusion constant *D* and addresses the probability that a given site *x* has never been visited by any walker up to time *t*. Equivalently, one could consider the site *x* to contain a particle with diffusion constant $D'=0$ and address the probability that the particle remains uninfected (i.e., unvisited). However, if the particle initially at position *x* is given a diffusion constant equal to that of the other particles in the system, i.e., $D'=D$, then the probability that it "persists" up to time *t* is the probability that it remains uninfected. This suggests that the number of uninfected particles is a function of the diffusion constants in the problem, with the limiting case $D'=0$ corresponding to the standard persistence problem. Clearly, in order to avoid contamination, remaining stationary is a more effective strategy than diffusing. Therefore, we anticipate that the fraction of uninfected walkers decays more rapidly than the fraction of uninfected (unvisited, i.e., persistent) sites. This faster decay of $U(t)$ is clearly observed in the simulations.

Given the similar decay of $P(t)$ in the $A\emptyset \leftrightarrow \emptyset A$ and $A+B\to\emptyset$ processes, these models make a sensible combination to study together. We discover that in both cases, $U(t) \sim [P(t)]^{\gamma}$ where $1<\gamma \lesssim 1.39$. In fact, the relation $U(t)=k[P(t)]^{\gamma}$, with k close to unity and $\gamma=1.39(1)$, describes the data rather accurately in both models over the whole range of times studied, although there are indications in both models that the asymptotic value of γ is smaller than 1.39. We discuss the $A\emptyset \leftrightarrow \emptyset A$ model in Sec. II and follow with the $A+B\to\emptyset$ reaction in Sec. III. We conclude in Sec. IV, with a discussion and summary of the results.

II. $A\emptyset \leftrightarrow \emptyset A$ MODEL

We consider a system of noninteracting, unbiased, diffusing walkers, with random initial conditions. Our numerics are performed on a 1D lattice of size $L=10^7$ with lattice-constant $a=1$ and periodic boundaries. At $t=0$, N random walkers are randomly distributed on the lattice with a maximum of one particle per site so that $U(0)=1$ and $P(0)=1-\rho_0$, where $\rho_0=\rho(0)=N/L$. Clearly, $P(0)\neq 1$ is a consequence of our lattice description and in the continuum limit $\rho_0\to 0$, where the mean distance between walkers is much larger than the lattice spacing $P(0)\to 1$. Therefore, in order to place the fraction of persistent sites and the fraction of uninfected walkers on an equal footing in our numerical study, we define

$$p(t)=\frac{P(t)}{P(0)} \tag{4}$$

so that both $U(0)$ and $p(0)$ are normalized to one. We now use $p(t)$ in the description of our numerical results.

Our model is updated using the direct method [33], i.e., at each computational step, a particle is picked at random and shifted with probability $D=1/2$ to a neighboring site, where D is the diffusion constant. For $t>0$, any site which is visited by a walker becomes nonpersistent for all remaining time *t*. Mutual infection occurs when two or more particles simultaneously occupy a single lattice site. For each jump made by a particle, time *t* is increased by $dt=1/N$, where N is the current number of particles in the system (which is constant in the present model). Our results are averaged over 100 independent runs.

Dimensional analysis demands that, on a *continuum*, the fraction of unvisited sites is a function of the dimensionless combination of parameters in the problem, i.e., $P(t)=f(\rho_0^2 Dt)$. Similarly, for the fraction of uninfected walkers $U(t)=g(\rho_0^2 Dt)$, where g is some other function. However, given that our numerical simulations take place on a discrete lattice, we expect the scaling of $P(t)$ and $U(t)$ as a function of $\rho_0^2 Dt$ to become strictly true only in the continuum limit $\rho_0\to 0$. In order to approximate this limit as closely as possible, we choose as our densities $\rho_0=0.007$, 0.005, and 0.003, so that $\rho_0 \ll 1$.

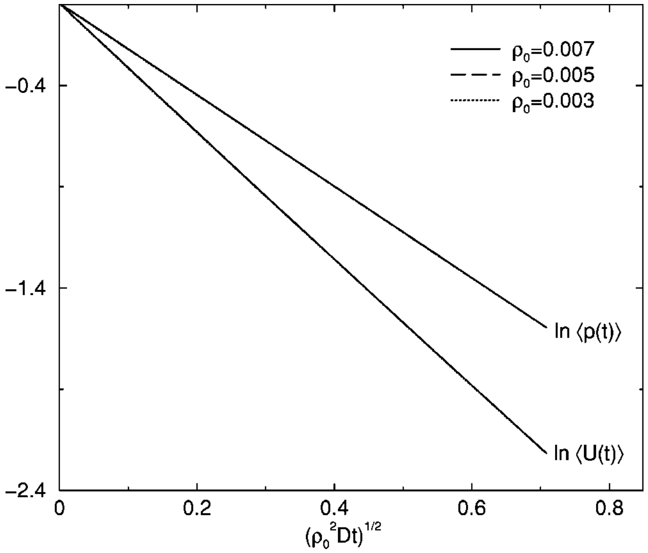

FIG. 1. Linear-log plot of the fraction of unvisited sites $p(t)$, and the fraction of uninfected walkers $U(t)$, as a function of $(\rho_0^2 Dt)^{1/2}$, for the $A\emptyset \leftrightarrow \emptyset A$ model.

The dimensionless combination of parameters, $\rho_0^2 Dt$, also defines a natural characteristic time scale in the problem, namely, $t^*=1/\rho_0^2 D$. Therefore, one expects to observe the asymptotic behavior of the system only after a time $t \gg t^*$ has been reached. However, as far as the persistent properties are concerned, Eq. (1) is an exact relation in the continuum limit for all *t*. Therefore, we expect to observe $P(t)$ approaching asymptopia very quickly in our numerical data. Clearly, in the discrete description of the model used in our simulations, Eq. (1) breaks down at the earliest times since, for very small *t*, lattice effects play a dominant role. Equation (2), however, which holds for $U(t)$ is a truly asymptotic relation. Unfortunately, the extremely fast decay of $U(t)$ leads to small-number effects at very early times in our simulations and we find the limit $t\gg t^*$ unattainable with good statistics. Therefore, we limit ourselves to studying the regime up to $t_{max} \simeq t^*/2$.

The fraction of unvisited sites decays, in the continuum limit, exactly as the stretched exponential form, Eq. (1) [7]. In order to make a direct comparison between the fraction of persistent sites $p(t)$ and the fraction of uninfected walkers $U(t)$, we present these two quantities on a linear-log plot in Fig. 1, using $(\rho_0^2 Dt)^{1/2}$ as abscissa. Figure 1 shows clearly that the data scale as a function of the dimensionless quantity $\rho_0^2 Dt$. $U(t)$ exhibits behavior characteristic of Eq. (2), with a faster decay rate than for the fraction of persistent sites, i.e., $\lambda > (16\rho_0^2 D/\pi)^{1/2}$. The data seem consistent with

$$U(t)\simeq K[p(t)]^{\gamma}, \tag{5}$$

with $K\simeq 1$ and $\gamma>1$. The log-log plot of $U(t)$ vs $p(t)$ in Fig. 2 indeed shows an almost linear relationship between

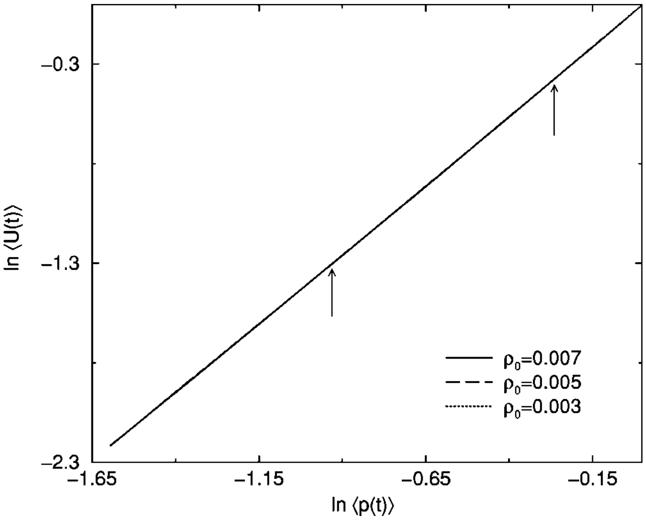

FIG. 2. Log-log plot of the fraction of uninfected walkers, $U(t)$, against the fraction of unvisited sites, $p(t)$, in the $A\varnothing \leftrightarrow \varnothing A$ model.

$\ln U(t)$ and $\ln p(t)$. In order to evaluate γ and K, we performed a linear regression on the data in Fig. 2 in the region indicated by the arrows, thereby avoiding initial transients associated with our lattice description, and the onset of statistical fluctuations between different runs of the data. We summarize our results for γ and K in Table I.

The data are sufficiently good to merit closer inspection. To check how far our numerics probe the asymptotic regime, we plot $-(\rho_0^2 Dt)^{-1/2}\ln p(t)$ and $-(\rho_0^2 Dt)^{-1/2}\ln U(t)$ against $\ln(\rho_0^2 Dt)$ in Fig. 3. The nature of this plot greatly expands the early time regime. Whereas the data collapse for different densities is excellent in Fig. 1, a systematic splitting of the curves as a function of ρ_0 is now clearly evident. This feature is a consequence of the discreteness of the lattice. In fact, the behavior of $p(t)$ is well described, for all t, by an exact lattice calculation [26] which predicts

$$p(t) = \exp[-\rho_0 a f(Dt/a^2)], \tag{6}$$

where a is the lattice spacing. The function $f(x)$ has the limiting behavior $f(x) \to 2x$ for $x \to 0$ and $f(x) \to (16x/\pi)^{1/2}$ for $x \to \infty$, the latter reproducing the continuum limit result, Eq. (1). Indeed, the structure of Eq. (6) shows that the continuum limit $a \to 0$ at fixed t, and the asymptotic limit $t \to \infty$ at fixed a are the same. Equation (6) also shows that the quantity $-(\rho_0^2 Dt)^{-1/2}\ln p(t)$ is, for fixed a and D, a function of t only, i.e., independent of ρ_0. For small t it behaves as $t^{1/2}$, accounting for the small-t behavior in Fig. 3. Plotting against $\ln(\rho_0^2 Dt)$ displaces the curves for different densities in the horizontal direction, as is clear in the figure. The whole region where the curves are split in Fig. 3 corresponds to the top left-hand corner of Fig. 1. If the data are plotted against $\ln t$ in Fig. 3, instead of $\ln(\rho_0^2 Dt)$, the data for different densities recollapse, but the onset of the continuum limit at late times is not as clear.

TABLE I. Numerical values of γ and K, where $U(t) \simeq K[p(t)]^{\gamma}$, for various ρ_0, in the $A\varnothing \leftrightarrow \varnothing A$ model.

ρ_0	γ	K
0.007	1.391(5)	0.990(3)
0.005	1.391(5)	0.991(3)
0.003	1.391(5)	0.991(3)

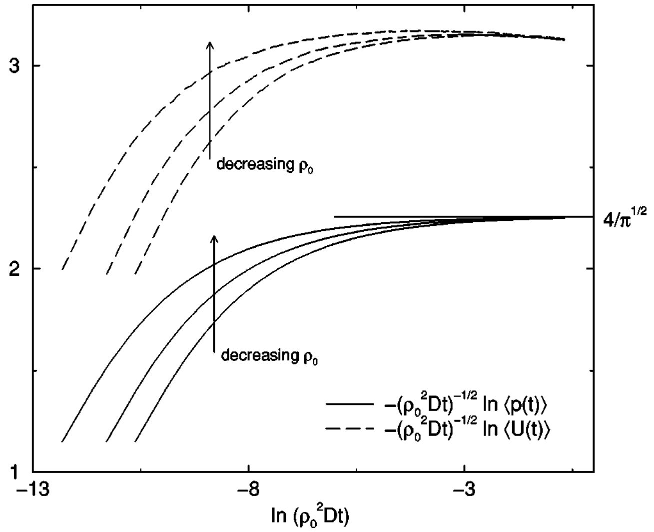

FIG. 3. $-(\rho_0^2 Dt)^{-1/2}\ln p(t)$ and $-(\rho_0^2 Dt)^{-1/2}\ln U(t)$ plotted against $\ln(\rho_0^2 Dt)$ for $\rho_0 = 0.007,\ 0.005$, and 0.003.

For large t, where the curves for the different densities converge in these scaled variables, the persistence data clearly approach the limiting value $4/\sqrt{\pi}$, consistent with the continuum limit displayed in Eq. (1). By contrast, Fig. 3 shows that all of our data for $U(t)$ is merely preasymptotic. The data reach a maximum and then appear to monotonically decrease in the region where the different curves converge (the continuum limit). Our results are consistent with the recent simulations of Mehra and Grassberger [27] who probe $U(t)$ to much smaller values than achieved here. The slow approach to asymptopia for $U(t)$ is not understood and, although our numerics seem roughly consistent with Eq. (2), a numerical determination of λ appears to be a hard problem. In the light of Fig. 3, Fig. 2 and its associated results (Table I) are somewhat misleading. Clearly, over the times covered by our simulations, γ cannot be a constant since λ is not.

In Fig. 4 we plot the effective exponent $\gamma(t) = \ln U(t)/\ln p(t)$ directly as a function of time. The data show that γ decreases monotonically with time even in the "continuum regime" where the curves for different densities converge. For $t \to \infty$, γ must tend to a constant if Eq. (2) is correct. We note that since the fraction of uninfected walkers decreases more rapidly than the fraction of persistent sites, $\gamma > 1$. Therefore, if the trend in the data continues, we can bound γ by $1 < \gamma < 1.39$. Asymptotically, $U(t) \sim [P(t)]^{\gamma}$ for some fixed γ where $1 < \gamma < 1.39$.

Let us consider once again the probability of being uninfected of a particle at site x, with diffusion constant D', and diffusing particles to its left and right with diffusion constant D. Then, we suggest that the fraction of uninfected walkers can be expressed as

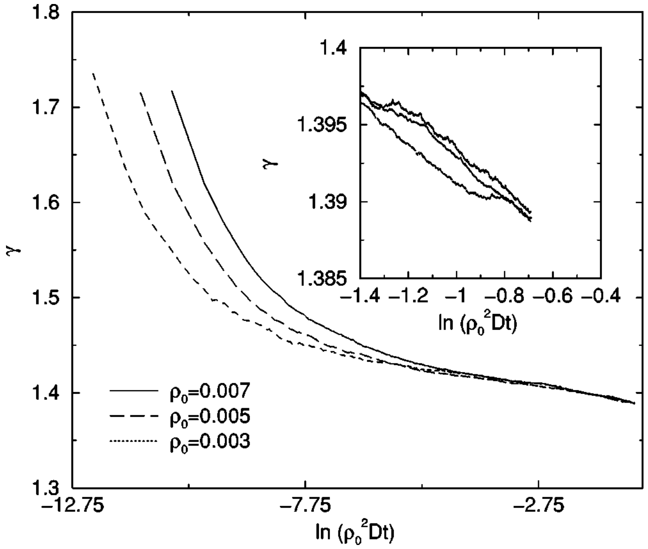

FIG. 4. Plot of γ vs $\ln(\rho_0^2 Dt)$, where $\gamma = \ln U(t)/\ln p(t)$, for the $A\emptyset \leftrightarrow \emptyset A$ model. Inset: blow up of the right-hand part of the plot.

$$U(t) \sim \exp[-(\rho_0^2 Dt)^{1/2} F(D'/D)] \tag{7}$$

for large t, consistent with the result of Bramson and Lebowitz [25] [Eq. (2)]. $F(x)$ is some unknown function of the ratio of the diffusion constants in the system. For $D' = 0$, the problem reduces to the familiar study of unvisited sites. Therefore, $F(0) = (16/\pi)^{1/2}$ and $U(t) = P(t)$. The case $D' = D$ corresponds to equally mobile particles, and the particle at site x becomes infected more quickly than if it had remained stationary. In this case, $F(1) = \gamma(16/\pi)^{1/2}$ giving $U(t) \sim [P(t)]^{\gamma}$ where, according to our numerics, $1 < \gamma < 1.39$. Although we have explored the values $D' = 0$ and $D' = D$ only, we conjecture that Eq. (7) holds for any ratio of the diffusion constants.

III. $A + B \to \emptyset$ MODEL

We now address the fraction of uninfected walkers $U(t)$ in the two-species annihilation model. Our simulations are performed on a 1D lattice of size $L = 10^7$ with periodic boundary conditions. At $t = 0$, exactly equal numbers, $N_A(0) = N_B(0)$, of A and B particles are randomly distributed on the lattice with a maximum of one particle per site, such that at large scales, both densities $\rho_A(0)$ and $\rho_B(0)$ are initially homogeneous. We define $\rho_A(0) = \rho_B(0) = N_A(0)/L$ and $\rho_0 = \rho_A(0) + \rho_B(0)$. Both species are also given the same diffusion constant $D_A = D_B = D = 1/2$. Our model is then updated in the same manner as for the $A\emptyset \leftrightarrow \emptyset A$ model described in Sec. II, i.e., $dt = 1/N(t)$, where $N(t) = N_A(t) + N_B(t)$, is the total current number of particles in the system. Infection occurs when two or more particles occupy a single lattice site, but we impose an instantaneous reaction $(A + B \to \emptyset)$ so that each lattice contains only one type of particle.

It is well known that in the two-species annihilation model in one dimension, there is an effective repulsion between the A and B particles that favors segregation into single species domains [5,28,32]. This domain coarsening leads to early time transients for both $P(t)$ and $U(t)$. Although using a high-initial density serves to accelerate the system's progression into the asymptotic regime, in doing so, not only is the continuum limit obscured, but we also find that the extremely fast decay of both $P(t)$ and $U(t)$ leads to small-number effects at very early times. Therefore we choose $\rho_0 \ll 1$. For consistency, we select our parameters to be the same as those studied in the $A\emptyset \leftrightarrow \emptyset A$ process, i.e., $\rho_0 = 0.007,\ 0.005$, and 0.003, and $\rho_0^2 Dt_{max} \simeq 0.5$ where t_{max} is the maximum number of time steps in the simulations. All the results are averaged over 100 runs.

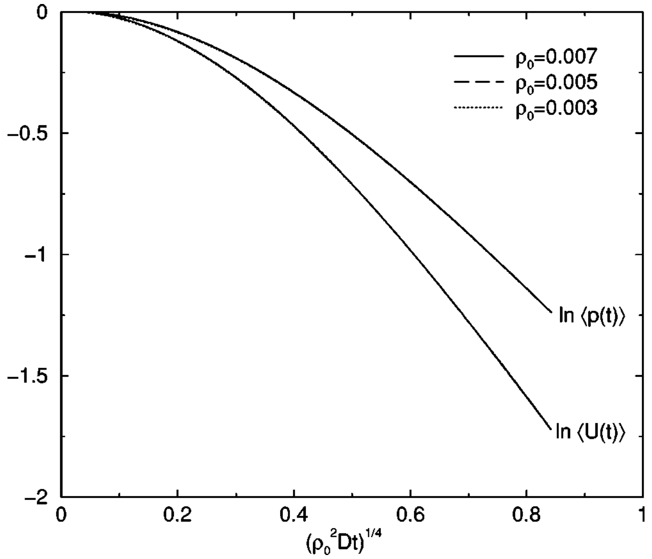

FIG. 5. Linear-log plot of the fraction of unvisited sites $p(t)$, and the fraction of uninfected walkers $U(t)$, as a function of $\rho_0^2 Dt$, for the $A + B \to \emptyset$ model.

The format of our analysis is much the same as in the preceding section. We observe that an algebraic relation $U(t) \sim [p(t)]^{\delta}$ between $U(t)$ and $p(t)$ also holds approximately in this case over the range of times studied. Surprisingly, δ seems to have a similar value to γ in the $A\emptyset \leftrightarrow \emptyset A$ model, i.e., $1 < \delta \lesssim 1.39$. For the two-species annihilation process we suggested in our earlier paper, on the basis of a toy model, that the fraction of unvisited sites decays asymptotically according to the stretched exponential form, Eq. (3) [7]. In the manner of Fig. 1, to make a direct comparison between the fraction of persistent sites $P(t)$, and the fraction of uninfected walkers $U(t)$, we present these two quantities on a linear-log plot in Fig. 5, this time using $(\rho_0^2 Dt)^{1/4}$ as abscissa, as suggested by Eq. (3). Figure 5 shows clearly that the data scale excellently as a function of the dimensionless quantity $\rho_0^2 Dt$. We appreciate, however, that the stretched exponential behavior of Eq. (3) is not particularly well realized. This may be due to the prevalence of large initial transients (we presented more convincing evidence of the exponential behavior in our earlier paper [7]). In particular, the asymptotic behavior represented in Eq. (3) is only expected for large values of $\rho_0^2 Dt$, unlike Eq. (1) for $P(t)$, which we have shown holds for all t on the continuum [7]. Nonetheless, even in the regime we have studied, it is clear from Fig.

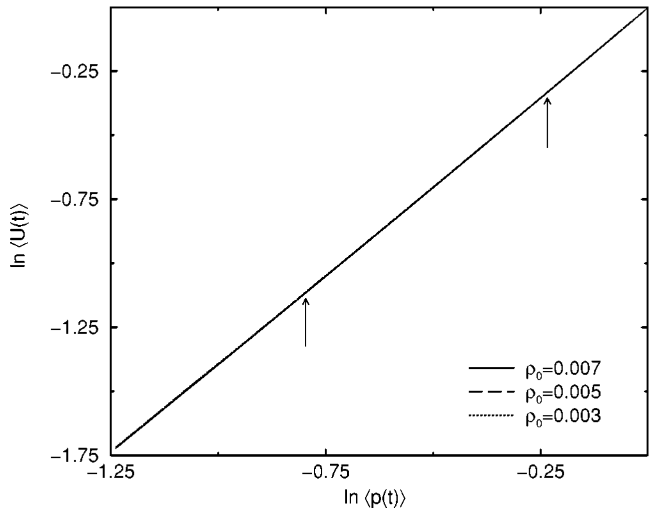

FIG. 6. Log-log plot of the fraction of uninfected walkers, $U(t)$, against the fraction of unvisited sites $p(t)$ in the $A+B\rightarrow\emptyset$ model.

5 that the fraction of uninfected walkers has a very similar form of decay to the fraction of unvisited sites, but with a faster decay rate. In Fig. 6, therefore, we plot $\ln U(t)$ against $\ln p(t)$ to see whether there is a simple relationship between $U(t)$ and $p(t)$.

The approximate linearity of the data in Fig. 6 suggests $U(t) \sim k[p(t)]^{\delta}$. To determine δ and k, we performed a linear regression on the data in the region indicated by the arrows, thereby avoiding initial transients associated with our lattice description, and the onset of statistical fluctuations between different runs of the data. We summarize our results for δ and k in Table II. Notice that δ and k agree remarkably well with γ and K recorded in Table I for the $A\emptyset \leftrightarrow \emptyset A$ process.

In analogy to Fig. 3, we plot $(\rho_0^2 Dt)^{-1/4} \ln p(t)$ and $(\rho_0^2 Dt)^{-1/4} \ln U(t)$ against $\ln(\rho_0^2 Dt)$ in Fig. 7. Early-time lattice effects, where the different densities do not overlap, are evident in Fig. 7. It is possible that all of our data for $p(t)$ and $U(t)$ are within the preasymptotic regime, since the $p(t)$ curve has not yet saturated to the late-time asymptotic values implied by Eq. (3). Note, however, that if the curves in Fig. 7 were asymptotically linearly increasing, rather than eventually saturating to a constant, the corresponding forms for $p(t)$ and $U(t)$ would be $\exp[-C(\rho_0^2 Dt)^{1/4} \ln(\rho_0^2 Dt)]$, with C a constant (different for p and U). This form differs only by the logarithmic term from that predicted in [7] on the basis of a toy model. It should be noted that neither our present data nor that presented in [7] can definitively rule out such logarithmic corrections, and the asympotic form of $p(t)$, as well as $U(t)$, cannot be regarded as being definitively established. Throughout the preasymptotic regime, however, the relation $U(t) = K[p(t)]^{\delta}$, with $\delta = 1.39(1)$, holds rather well. In Fig. 8 we plot the effective exponent $\delta(t) = \ln U(t)/\ln p(t)$ directly as a function of $\ln(\rho_0^2 Dt)$.

TABLE II. Numerical values of δ and k, where $U(t) \simeq k[p(t)]^{\delta}$, for various ρ_0, in the $A+B\rightarrow\emptyset$ model.

ρ_0	δ	k
0.007	1.390(5)	0.990(3)
0.005	1.390(5)	0.993(3)
0.003	1.386(5)	0.991(3)

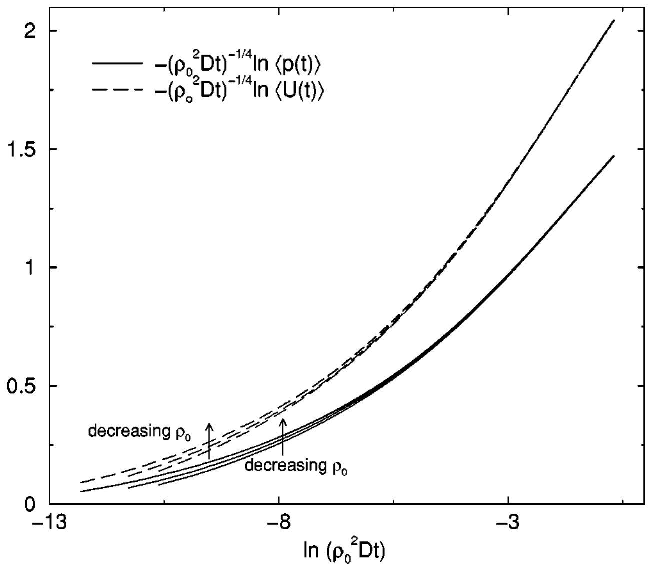

FIG. 7. $-(\rho_0^2 Dt)^{-1/4} \ln p(t)$ and $-(\rho_0^2 Dt)^{-1/4} \ln U(t)$ plotted against $\ln(\rho_0^2 Dt)$ for $\rho_0 = 0.007$, 0.005, and 0.003.

Figure 8 is remarkably similar to Fig. 4 for the $A\emptyset \leftrightarrow \emptyset A$ process. Indeed, except for the earliest times, the corresponding curves differ by less by 1% everywhere. In a similar way we can bound the asymptotic δ (assuming that the late-time trend evident in Fig. 8 continues) by $1 < \delta < 1.39$. Since we have argued elsewhere [7], using a toy model, that the asymptotic behavior of $P(t)$ is described by Eq. (3), we conjecture that the asymptotics of $U(t)$ are given by

$$U(t) \sim \exp[-\delta C(\rho_0^2 Dt)^{1/4}], \tag{8}$$

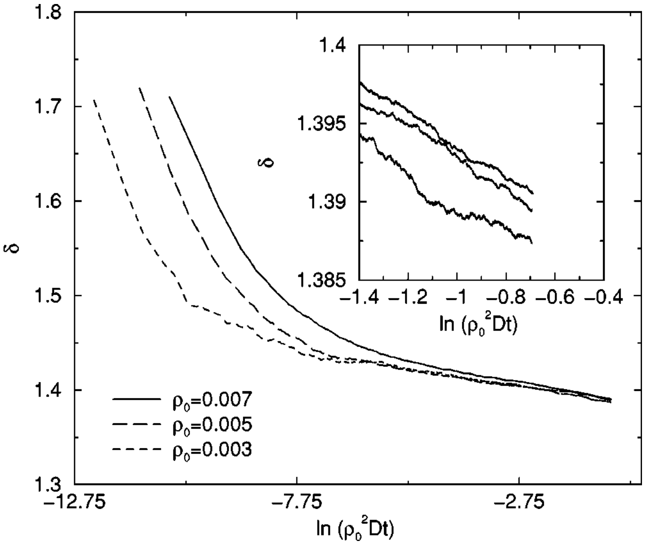

FIG. 8. Plot of δ vs $\ln t$, where $\delta = \ln U(t)/\ln p(t)$, for the $A+B\rightarrow\emptyset$ model. Inset: blow up of the right-hand part of the data.

where C is a constant and $1<\delta<1.39$. For the $A\varnothing\leftrightarrow\varnothing A$ model, we expressed [Eq. (7)] $U(t)$ in terms of the ratio D'/D, where D' is the diffusion constant of a tagged particle, D is the diffusion constant of the other particles, and $U(t)$ is the probability that the tagged particle remains uninfected at time t. The analogous form for the $A+B\rightarrow\varnothing$ process is naturally

$$U(t)\sim \exp[-(\rho_0^2 Dt)^{1/4} G(D'/D)], \tag{9}$$

where $G(x)$ is some unknown function. As before, the case $D'=0$ corresponds to the problem of unvisited sites, so that $G(0)=C$ and $U(t)=P(t)$. For $D'=D$, the case studied above, our conjecture implies $G(1)=\delta C$. However, we stress that a logarithmic correction of the form discussed above cannot be ruled out by the data. The main motivation for Eq. (9) is the toy model presented in [7], applied to the case $D'=0$.

IV. DISCUSSION AND SUMMARY

In this paper, we have studied two distinct models: (i) a one-dimensional system of noninteracting, unbiased diffusing walkers $A\varnothing\leftrightarrow\varnothing A$, with random initial conditions and (ii) the $A+B\rightarrow\varnothing$ diffusion limited process in one dimension with equal initial densities of A and B particles distributed homogeneously at $t=0$. We have reduced the familiar study of persistent sites $P(t)$, to a limiting case in the study of uninfected walkers and shown numerically that in both models

$$U(t)\sim[P(t)]^{\gamma}, \tag{10}$$

where $\gamma\simeq 1.39$ for the time regimes covered by our data and $1<\gamma<1.39$ for asymptotically large time. The common feature of the $A\varnothing\leftrightarrow\varnothing A$ and $A+B\rightarrow\varnothing$ processes is that the fraction of unvisited sites decays according to a stretched exponential (possibly with logarithmic corrections for $A+B\rightarrow\varnothing$). In our earlier paper [7], we argued heuristically, on the basis of a toy model, that if the density of walkers decays asymptotically as $t^{-\alpha}$, then the fraction of unvisited sites $P(t)$ decays with a stretched exponential form if $\alpha<1/2$. We showed that

$$P(t)\sim \exp-[A(\alpha)\rho_0^2 Dt]^{(1/2)-\alpha}, \quad \alpha<1/2, \tag{11}$$

where A is some function of α. This result is consistent with the results for models studied here. Remarkably, Eq. (10) seems to hold for both the $A\varnothing\leftrightarrow\varnothing A$ and the $A+B\rightarrow\varnothing$ processes, with approximately the same bounds on γ. Furthermore, the effective time-dependent value of γ is very similar for both models (see Figs. 4 and 8). An obvious goal for the future is to try to obtain a theoretical understanding of this simple relationship, and test it against other models with $\alpha<1/2$.

Even for the simple case $\alpha=0$ (the $A\varnothing\leftrightarrow\varnothing A$ model) studied in this paper, an analytical calculation of $U(t)$ is a challenging problem. For a system consisting of just three particles, the probability that the central particle has not been crossed by either of the outer particles can be calculated exactly [34], and is found to decay as $t^{-3/2}$. The problem is solved by mapping it onto diffusion in two dimensions with absorbing boundaries on a wedge [34]. For $N>3$ particles, the problem maps onto diffusion in $N-1$ dimensions with absorbing boundaries on a hyperwedge. This problem has so far proved intractable.

For both models studied here, we expressed the probability that a particle at site x is uninfected in terms of its diffusion constant D' and the diffusion constant D of the other particles [Eqs. (7) and (9)], noting that $P(t)$ corresponds to the limit $D'=0$. Using Eq. (11), we can generalize this description to any system of randomly diffusing particles for which the density decays as $\sim t^{-\alpha}$ with $\alpha<1/2$. We conjecture that, in this case, the fraction of uninfected walkers can be expressed, asymptotically, as

$$U(t)\sim \exp[-(\rho_0^2 Dt)^{(1/2)-\alpha} R(D'/D,\alpha)], \tag{12}$$

with $U(D'=0)=P(t)$ and R some unknown function. Clearly, there is much scope to test the validity of this form since we have only considered $\alpha=0,1/4$ and $D'=0,D$. Note that for systems with $\alpha>1/2$, our toy model predicts [7] that $P(t)$ approaches a nonzero constant, while $U(t)$ certainly behaves differently since it is bounded above by $\rho(t)$ and must therefore vanish for large t. More interesting, however, is the borderline case where the particle density falls off like $t^{-1/2}$. In this case, our toy model predicts that $P(t)$ [7] decays as a power law, $P(t)\sim t^{-\theta}$. A system where such behavior is observed is the well studied q-state Potts model [9,10,12,13,20]. The results of our study of uninfected walkers in this model follows in a separate paper [35].

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